**Many-Body Harmonic Oscillators again**

And some examples…

**N Harmonic Oscillators on a Ring (periodic BC)**

Here’s a more explicit example, in 1D. This is also how we’d do it in QFT. We’ll end up with a result like before. Say we have N harmonic oscillators with ‘lattice spacing’ a and total length L.

A picture containing art

Description automatically generated with medium confidence

The Hamiltonian of our system is:

where K = Mionω2. We can work out the solution using the Heisenberg picture, i.e., we’ll look at how the operators develop in time. We’ll start out with the Lagrangian:

and solve for the equations of the oscillators. We can for instance use the E-L equations, or whatever to find:

(changing index from k → ℓ in last line) We can write this in matrix notation,



where,



Now we wish to diagaonalize the ‘spatial’ part, so we construct the ‘eigenfunction’ expansion:



In matrix form, this will look like,



Recalling the previous file, this means Aℓm will constitute the ‘normal’ modes, **A**(j).



And we can invert it: Qℓ(t) = Σm A-1ℓmqm(t), and plug this into the equation…we will see of course that the matrix A is constructed so that its rows are simply the eigenvectors of the spatial part so that, when the matrix multiplication is expanded as instructed above, it just turns out to be a sum over modes times that mode’s time development. So filling this ansatz into our equation, we have:



And we’ll want to find an A such that:



which implies,



which states (see Matrix file) that the column vectors of A are the eigenvectors of V. So to determine the eigenvectors of the matrix V, we form (calling the eigenvalues ωn2):



Let’s try something of the form A = eλl, and n is just a spectator variable for now. This exponential ansatz is analogous to guessing eλx for linear differential equations. Before we plug it in, let’s note that our result needs to be periodic in ℓ, such that A(ℓ = N) = A(ℓ = 0), since we have periodic boundary conditions. So,



In practice n is restricted to between [0, N) as n out of this range will simply reproduce the values within the range. Actually, the customary range to put n in is [-N/2, N/2). This is called the Brillouin Zone (BZ). So anyway, now we have:



Before we proceed, should note that A is complex valued. So in order for the qn to be real, the Qn must also be complex valued. And we need,



But then can change n → -n in the summation in right side term, and have:



So that’s a restriction on the d.o.f. of our Q variables. This is good because now the total number of d.o.f. of the q’s (N) is the same as the number of d.o.f. of the Q’s (N/2·2 = N), where N/2 is for only positive n’s being independent, and the 2 is for the real and imaginary parts of Q. Anway, plugging this into our equation,



So this ansatz works, and we have our eigenvalues too. Going to clean it up a little bit. First I’ll add a normalization factor 1/√N to make A a unitary matrix, i.e., so that its column vectors are normalized.



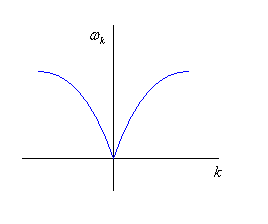
where km = 2πm/L is a wave-vector, and Rn = an is the position of the nth oscillator. We will note Amn = Anm. And the eigenvalues can be written as:



and so the eigenvectors and eigenvalues are:



which we can just recognize as an acoustic branch (in 1D and with no basis, we have no other excitations).



Note how for small k (and or small a) this reduces to ωn = ω|akn|. We ought to compare this result to the one obtained in the continuum case. In the continuum-ized version we obtain the same long wavelength result. So we should expect that a continuumized version of the theory would correctly reproduce the long wavelength dynamics – but not the short wavelength dynamics. **And observe the acoustic spectrum – present b/c of n.n. interaction.** So we have



and the solution to this equation is trivial:



The fact that this operator evolves harmonically, with eigenvalue ωℓ means that the Hamiltonian has excitations ωℓ (see QM Foundations/Heisenberg Picture file). So now already know the energies levels of the Hamiltonian. They are just integer linear combinations of the ωℓ, i.e., E = Σℓnℓωℓ. Actually there is a ½ in there. We wouldn’t know that per se´ without a little more work, but that only adjusts the ground state energy, which we don’t really care about anyway.



But anyway, we’ll go on to work out an expression for q and p in terms of the creation and annihilation operators. So that the general solution of the equation of motion is:



Of course, as discussed in the Foundations folder/Heisenberg Time Development, we can identify the creation/annihilation operators as being proportional to:



using the canonical commutation relations:



we can determine that:



And using this commutation relation we can normalize the creation / annihilation operators so that:



So we have that:



I think that I will use the BZ that ranges from -N/2,…,N/2; that way I can say:



And we can write:



(because the energy is even in the index m)



This no doubt can be generally written:



that is, k runs over a range of 2π/a (usually chosen as between –π/a and π/a ) in increments of 2π/L. We can work out the momentum operator too:



So,



Now let’s look at the Hamiltonian,



Let’s do the p sum first,



Can set t = 0, since t shouldn’t matter (t might matter for different parts of H, but the total H)



So there’s that. Next, let’s look at, at t = 0:



and finally,



And now computing H,



Now trying to simplify some. Let’s use aa† = 1 + a†a,



Combining similar terms,



OMG. Apropos the e-ik\_ma factors, we can break it down into cos(kma) + isin(kma). And we can ignore the sin() part, as this will be odd in m, and make the sum go to zero (also use fact that

a-mam = ama-m). Then we have:



Now use ωm2 = 2ω2(1-coskma),



So our Hamiltonian is, finally:



And this confirms our supposition about the energy eigenvalues up above.

**Wavefunctions**

So note that there are N (labelled by the index m) normal modes



which specify a distinct initial displacement of the masses which upon release will execute a repeating motion. The initial of the ℓth particle in the mth normal mode is given by Nm(ℓ). Each of these N normal modes has a unique (unless H is degenerate?) frequency of repetition/energy: ωm. The Hamiltonian above tells us that we can break any excitation of the system into a sum of normal mode excitations. The creation operator in the Hamiltonian creates one excitation of the system in its particular normal mode. We can think of this excitation – the resulting periodic movement of the system corresponding to that normal mode – as a particle (phonon/photon, etc.). The eigenfunctions live in the N dimensional HS. They are,



If we were to apply <q1q2…qN|to the back of each side of this equation we would obtain the differential equation for the particular excited state. They could be determined from



and this would be just an N-variable wavefunction, giving the ‘vector’ that diagonalizes the Hamiltonian. Note that H lives in the HS spanned by the basis vectors |q1q2…qN>. This could be theoretically solved simply by making the appropriate change of independent variables that diagonalizes the position dependent matrix in xℓ And then this would reduce to a sum of harmonic oscillator PDE’s, which could be solved via the standard methods, taking into account the boundary conditions that it is periodic in every variable with period L. So let’s do a little along these lines. Start with H,



We can write this as:



where V is defined as before,



since,



where we’ve made changes of variables (index) and liberally taken advantage of the fact that qi+N = qi. Now we want to diagonalize V. And of course we already did this. So we do the coordinate transformation (these are coordinates, not operators now)



(the sum extends over the BZ) where,



and km = 2πm/L is a wave-vector, and Rn = an is the position of the nth oscillator. We will note Amn = Anm. And then,



So now we have:



where I’m trying to be careful about the limits of summation. Let’s also work out the derivative. So,



Taking advantage of the fact that Amn is a unitary matrix, so that A-1mn = A\*nm, we have:



So H is now,



One more thing. We’ll note that: Q-n = Qn\*, and so could write this as:



So now we have n copies of a single ODE, but in a complex variable Qn. Let’s now change variables again, to Xn = ReQn and Yn = ImQn. Then:



So we can write H as:



So now we have an H which constitutes 2(N/2) = N copies of a harmonic oscillator. Now observe that since Q-n = Qn\*, we can say Yn>0 = Xn<0, and write (abetted by fact that ωn2 = ω-n2)



But maybe I’ll write this instead as:



where kn = 2πni/N, as we’ll recall. So this is our H in X = ReQ space. So we can immediately write down the wavefunction and energies,



Of course this is given in terms of the transformed coordinates Xn = ReQn. To put in terms of qn we’ll recall:



and so since qn are all real,



An eigenfunction of the Hamiltonian in the truest sense ought to give us the probability that the set of harmonic oscillators assumes a particular configuration. A particular configuration is a specification of N variables defining the particular eigenvalue that the operator at la, qℓ, assumes. Thus it ought to really be an N-valued function as we see above – to give the probability of a particular field configuration. On the other hand, there are these single…multiple particle wavefunctions that bear most resemblance to the simply normal modes, which I will look into below. These clearly are not wavefunctions of the Hamiltonian.



so we form



Then using the FFE for qℓ, we commute these two operators, arriving in the process at the expression



We recognize this as just one of the normal modes, and we interpret



that is, that the operator at l acting on the ground state creates a particle excitation at l. Two particle eigenfunctions can be obtained similarly



This results in



which is the symmetrized product of two single particle eigenfunctions, as we should expect.

**N Clamped Harmonic Oscillators**

Let’s do the same as above, but with different boundary conditions.

Starting with Lagrangian again,

We have the same equations of motion…

We can write this in matrix notation as

 Einstein summation convention

where



Now we wish to diagaonalize the ‘spatial’ part, so we construct the eigenfunction expansion qℓ(t) = Aℓm(t) → Qℓ(t) = A-1ℓmqm(t) and plug this into the equation…



So we want A to diagonalize the potential V, such that



which implies that



which states (see Matrix file) that the column vectors of A are the eigenvectors of V. So to determine the eigenvectors of the matrix V, we form



Whatever trial solution we use, it must satisfy the clamped boundar conditions in the row index, so will try…



as a trial solution. And this produces (we can consider just the imaginary part of the preceeding analysis for the case of periodic boundary conditions)



The prefactor to the eigenfunction is chosen so that A is orthogonal – as it should be. So we have



and the solution to this equation is trivial



So that the general solution of the equation of motion is



Of course, we can identify the creation/annihilation operators as being proportional to



this time, since Q’s are real, we don’t have the same issue we did last time. using the canonical commutation relations



we can determine that



And using this commutation relation we can normalize the creation / annihilation operators so that



So we have that



And we can write



If we plug the free field expansion into the expression for the Hamiltonian, we ought to get

